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The equilibrium of a rigid body on a plane with anisotropic dry friction☆

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ABSTRACT

The problem of the conditions for the static equilibrium of a body, resting on a rough plane at one, two or three points, is considered. It is assumed that an arbitrary system of active forces is applied to the body, while the friction on the rough supporting plane is anisotropic. This model generalizes the well-known isotropic model of Coulomb dry friction. Explicit analytic formulae, which express the necessary and sufficient conditions for static equilibrium, are obtained. The investigation procedure uses the idea of an anisotropic force of static friction, which enables analytical results for the equilibrium conditions to be obtained more easily.

1. Formulation of the equilibrium problem, description of the model of anisotropic friction and fundamental equations

Suppose Oxyz is a fixed system of coordinates. Consider a rigid body, resting on a plane Oxy at its points: $A_1, \ldots, A_n$. We will denote the radius vector of the point A_k by $\mathbf{r}_k = (x_k, y_k, 0)^T$. Here and everywhere below, unless otherwise stated, $k = 1, 2, \ldots, n$. In Fig. 1 we show the supporting plane Oxy (as seen from above from the positive direction of the Oz axis), on which we show one point of support A_k. The reaction of the plane at the point A_k consists of the normal component $\mathbf{N}_k$, directed along the Oz axis, and the tangential component $\mathbf{F}_k$ (the friction force), lying in the Oxy plane. Suppose an arbitrary system of active forces, having a principal vector $\mathbf{F} = (F_x, F_y, F_z)^T$ and a principal moment about the point O $\mathbf{M}_O = (M_x, M_y, M_z)^T$ is applied to the body. It is required to determine the conditions imposed on the quantities $\mathbf{F}$ and $\mathbf{M}_O$, the coordinates of the points A_k and the characteristics of the friction at the points A_k, for which the reactions $\mathbf{N}_k = (0, 0, N_k)^T$, $\mathbf{F}_k = (F_{kx}, F_{ky}, 0)^T$ are such that the following conditions of static equilibrium of the body are satisfied

$$\mathbf{F} + \sum_k (\mathbf{N}_k + \mathbf{F}_k) = 0, \quad \mathbf{M}_O + \sum_k [\mathbf{r}_k \times (\mathbf{N}_k + \mathbf{F}_k)] = 0 \tag{1.1}$$

Moreover, the conditions that the projection of the normal reaction onto the vertical ($N_k \geq 0$) must be non-negative and the corresponding inequalities for the friction forces at rest $\mathbf{F}_k$ for anisotropic dry friction must be satisfied.

Following the well-known approach,[1] we will write a model for the anisotropic dry friction which generalizes the usual Coulomb's law (isotropic dry friction).

Suppose the contact point A_k has acquired a velocity υ_k in the Oxy plane, directed at an angle θ to the positive Ox axis (Fig. 1). Then the anisotropic slip friction force is given by the formula

$$\mathbf{F}_{\rm fr} = -N_k \mathbf{\Phi}_O \boldsymbol{\upsilon}_k / |\boldsymbol{\upsilon}_k| \tag{1.2}$$

where $\Phi_O = ||f_{ij}||$ is a 2×2 matrix of the friction tensor, assumed positive definite, since the power of the force $\mathbf{F}_{\rm fr}$ for any velocity υ must be negative, i.e., $(\upsilon^T \Phi_O \upsilon) > 0$. Consequently, it is necessary for the conditions $f_{11} > 0, f_{22} > 0$, $\Delta = f_{11}f_{22} - f_{12}f_{21} > 0$ to be satisfied.

Note that the necessary and sufficient conditions for the matrix Φ_O to be positive definite are the inequalities

$$f_{11} > 0, \quad f_{22} > 0, \quad f_{11}f_{22} - (f_{12} + f_{21})^2/4 > 0$$

which are identical with those derived only when $f_{12} = f_{21}$.

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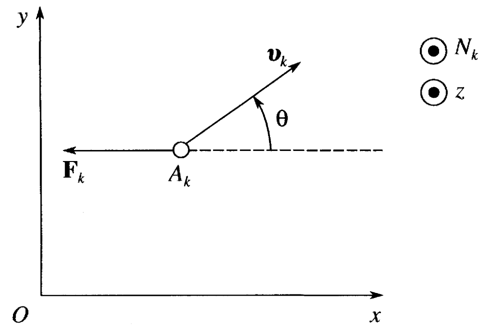

Fig. 1.

For the classical law of isotropic dry friction (Coulomb's law) we have $\Phi_O = \mathrm{fE}$, where E is the unit matrix and f is the coefficient of friction. Projecting the vector inequality (1.2) onto the O*x* and O*y* axis, we obtain

$$F_{\mathrm{fr},x} = -N_k(f_{11}\cos\theta + f_{12}\sin\theta), \quad F_{\mathrm{fr}y} = -N_k(f_{21}\cos\theta + f_{22}\sin\theta) \tag{1.3}$$

Formula (1.2) for the anisotropic friction force for motion (for initial motion) implies the presence of a corresponding anisotropic force of static friction F_k which is determined using the following principle.

The static friction force F_k, directed at an angle α to the positive Ox axis, does not exceed in modulus the modulus of the possible force of kinetic friction, which is also directed at an angle α to the Ox axis. Clearly, the corresponding possible initial slip point A_k occurs at an angle θ to the Ox axis, which is determined using formula (1.3) and the equalities

$$\mathrm{tg}\,\alpha = F_{\mathrm{fr}y}/F_{\mathrm{fr}x} = F_{ky}/F_{kx} \tag{1.4}$$

We then obtain

$$\mathrm{tg}\,\theta = (f_{11}\mathrm{tg}\,\alpha - f_{21})/(f_{22} - f_{12}\mathrm{tg}\,\alpha) \tag{1.5}$$

Substituting expression (1.5) into relations (1.3), we have

$$F_{\mathrm{fr}x}^2 + F_{\mathrm{fr}y}^2 = \Delta^2 N_k^2(1 + \mathrm{tg}^2\alpha)/(a^2 + b^2); \quad a = f_{22} - f_{12}\mathrm{tg}\,\alpha, \quad b = f_{11}\mathrm{tg}\,\alpha - f_{21}$$

Then, taking the last equality, the principle formulated above and relations (1.4) into account, we obtain the following inequalities for the components of the static friction forces F_k

$$(f_{12}F_{ky} - f_{22}F_{kx})^2 + (f_{21}F_{kx} - f_{11}F_{ky})^2 \le \xi_k^2 = \Delta^2 N_k^2; \quad \Delta = f_{11}f_{22} - f_{12}f_{21} > 0 \tag{1.6}$$

In exactly the same way we can determine the conditions for the static friction forces and for other laws of anisotropic sliding friction, which were presented previously in Ref. 2.

If the friction is orthotropic, i.e., the Ox and Oy axes are the principal directions of the matrix Φ_O, we have $f_{12} = f_{21} = 0$ and limit (1.6) takes the form

$$F_{kx}^2/(f_{11}N_k)^2 + F_{ky}^2/(f_{22}N_k)^2 \le 1$$

i.e., it is a friction ellipse. If the friction is isotropic (Coulomb's law), we have $f_{11} = f_{22} = f$, and we obtain a friction circle.

Thus, the problem in question is formulated as follows. It is required to determine the conditions, imposed on the quantities $F_x, F_y, F_z, M_x, M_y, M_z, (x_k, y_k)$ and the coefficients of friction f_{ij} (i, j = 1, 2), for which reactions $N_k = (0, 0, N_k)^T$, $F_k = (F_{ks}, F_{ky}, 0)^T$ exist such that the equilibrium equations (1.1) are satisfied and inequalities (1.6) and $N_k \ge 0$ are also satisfied, i.e., the reaction forces of the plane are admissible.

In this paper we will consider the cases n = 1, n = 2 and n = 3, i.e., the number of support points does not exceed three. These cases are statically definite for normal reactions and can be considered within the framework of the model of an absolutely rigid body. The case n > 3 was considered previously in Refs 3 and 4; in particular, the conditions of guaranteed equilibrium were investigated. It was shown[3] that the conditions of guaranteed equilibrium (for n > 3) reduce to investigating the equilibrium problem for support at any three of the specified *n* support points. Hence, the case n = 3 is decisive when investigating problems of the guaranteed equilibrium of a body with an arbitrary number of support points within the framework of the model of an absolutely rigid body.

Note that the problem considered in this paper is only concerned with finding the conditions for which equilibrium can be obtained. Problems of the stability of these equilibria are not considered here. These problems require the use of the dynamic equations of motion of a rigid body on a rough plane, and have been investigated in certain cases in Ref. 5.

2. Formulation and justification of the results in the case when n = 1

Suppose A_1 is the only point of support. Without loss of generality we will assume that it coincides with the origin of coordinates *O*, i.e., $x_1 = y_1 = 0$. Equilibrium equations (1.1) then reduce to the following

$$F_x + F_{1x} = 0, \quad F_y + F_{1y} = 0, \quad F_z + N_1 = 0, \quad M_x = M_y = M_z = 0 \tag{2.1}$$

Inequality (1.6) for the admissible reaction forces when k = 1 has the form

$$(f_{12}F_{1y} - f_{22}F_{1x})^2 + (f_{21}F_{1x} - f_{11}F_{1y})^2 \le \Delta^2 N_1^2, \quad N_1 = -F_z > 0 \tag{2.2}$$

We immediately obtain the following result from relations (2.1) and (2.2).

Assertion 1. For the static equilibrium of the body, supported at a single point A_1 on a plane with anisotropic dry friction, characterized by the 2 × 2 matrix $\Phi_O = ||f_{ij}||$, it is necessary and sufficient for the following conditions to be satisfied

$$M_{A_1} = 0, \quad F_z < 0, \quad (f_{12}F_y - f_{22}F_x)^2 + (f_{21}F_x - f_{11}F_y)^2 \le \Delta^2 F_z^2 \tag{2.3}$$

where M_{A_1} is the principal moment of the active forces about the point A_1 and $F = (F_x, F_y, F_z)^T$ is the principal vector of the active forces.

Corollary. *In the case of isotropic dry friction $F_{12} = F_{21} = 0$, $F_{11} = F_{22} = f$, and conditions (2.3) have the form*

$$M_{A_1} = 0, \quad F_z < 0, \quad F_x^2 + F_y^2 \le f^2 F_z^2 = f^2 N^2$$

i.e., *we obtain the usual Coulomb cone of friction.*

Example. A heavy point of mass m rests on a rough anisotropic inclined plane. Suppose φ_0 is the slope of the plane to the horizon and the system of coordinates Oxyz, in which the friction matrix Φ_O is specified, is such that the Ox axis is normal to the plane and the Ox axis makes an angle of ψ_0 with the straight line of greatest slope. Then

$$F_x = mg\sin\varphi_0\cos\psi_0, \quad F_y = mg\sin\varphi_0\sin\psi_0, \quad F_z = -mg\cos\varphi_0$$

and the equilibrium condition has the form

$$\operatorname{tg}\varphi_0 \le \Delta/\sqrt{a^2 + b^2}; \quad a = f_{12}\sin\psi_0 - f_{22}\cos\psi_0, \quad b = f_{21}\cos\psi_0 - f_{11}\sin\psi_0$$

In particular, for isotropic dry friction this condition takes the well-known form $\operatorname{tg}\varphi_0 \le f$, where f is the coefficient of friction.

A special case of this problem was investigated in Ref. 6, namely, the beginning of motion of a point mass along a plane with orthotropic friction ($f_{12} = f_{21} = 0$). The beginning of the motion is the first instant when the point leaves the state of static equilibrium. Violation of the last inequality in (2.3) also denotes the possibility of the beginning of motion of a point mass under the action of an active force F, since the first two conditions are necessarily satisfied (when $F_z < 0$).

Hence, for orthotropic friction breakdown of equilibrium of the point occurs when

$$f_{22}^2 F_x^2 + f_{11}^2 F_y^2 > f_{11}^2 f_{22}^2 F_z^2$$

If the value of F_z is fixed, the minimum force acting in the Oxy plane at an angle δ to the Ox axis and which disturbs the equilibrium of the point is given by the formula

$$F_{\min} = f_{11}f_{22}|F_z|/\sqrt{f_{22}^2\cos^2\delta + f_{11}^2\sin^2\delta}$$

This result was obtained in Ref. 6 by the limit equilibrium method, which goes back to Coulomb, Jellett and Zhukovskii and other classics of theoretical mechanics. The method is based on the assumption that the point begins to move from a state of rest (equilibrium). As a result of this assumption, the modulus and direction of the friction force become known, in accordance with formulae (1.3) and (1.4). One can then derive the conditions which the force F must satisfy in order that this motion can take place for any angle θ, which the vector of the velocity of possible slippage makes with the positive Ox axis. Non-satisfaction of these conditions for all angles θ leads to conditions of equilibrium. Strict application of this method for the problem being considered here is extremely lengthy and will not be done here.

3. Formulation and justification of the results for the case when n = 2

Suppose A_1 and A_2 are two points of support of the body on the plane, and A_1 coincides with the origin of coordinates while point A_2 has coordinates $x_2 = a\cos\alpha$, $y_2 = a\sin\alpha$, where a is the length of the section A_1A_2 and α is the angle which the vector $\overrightarrow{A_1A_2}$ makes with the positive Ox axis (Fig. 2). In Fig. 2 we also show the vector $\mathbf{F}_{xy}$, which is the projection of the principal vector $\mathbf{F}$ on the Oxy plane and ψ_0 is the angle which the vector $\mathbf{F}_{xy}$ makes with the vector $\overrightarrow{A_1A_2}$. All the angles are measured in an anticlockwise direction.

In this case equilibrium equations (1.1) have the form

$$F_{1x} + F_{2x} + F_x = 0, \quad F_{1y} + F_{2y} + F_y = 0, \quad N_1 + N_2 = -F_z \tag{3.1}$$

$$M_x + N_2 a\sin\alpha = 0, \quad M_y - N_2 a\cos\alpha = 0, \quad M_z + F_{2y}a\cos\alpha - F_{2x}a\sin\alpha = 0 \tag{3.2}$$

The first two equations of system (3.2) impose the following constrain on M_x and M_y

$$M_x\cos\alpha + M_y\sin\alpha = 0 \tag{3.3}$$

which denotes that there are no conditions which ensure rotation of the body about the A_1A_2 axis. Consequently, we obtain the following expressions for the normal reactions from Eqs (3.1) and (3.2)

$$N_1 = -F_z - M_y/(a\cos\alpha) > 0, \quad N_2 = M_y/(a\cos\alpha) > 0 \tag{3.4}$$

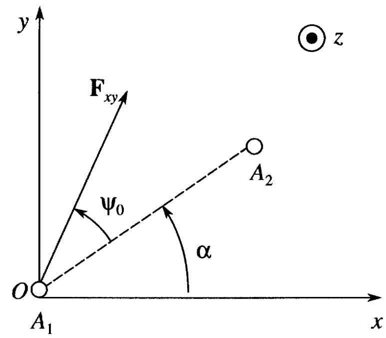

Fig. 2.

Hence, here the normal reactions, in a state of static equilibrium, depend only on the specified active forces, their moments and the geometrical parameters. Hence, we will henceforth assume that N_1 and N_2 are specified positive quantities defined by formulae (3.4), where $N_2 \geq N_1$, which does not violate the generality of the discussion. To solve the problem of the equilibrium conditions it is necessary, using equilibrium equations (3.1) and (3.2), to satisfy inequalities (1.6) with k = 1, 2 for static friction forces.

We will introduce the following notation

$$\begin{aligned}
&g(\varphi) = f_{22}\cos\varphi - f_{12}\sin\varphi, \quad h(\varphi) = f_{11}\sin\varphi - f_{21}\cos\varphi \\
&\sigma^2(\varphi) = g^2(\varphi) + h^2(\varphi), \quad \gamma = \alpha + \psi_0, \quad \Delta = f_{11}f_{22} - f_{12}f_{21} \\
&\kappa = g(\alpha)g(\gamma) + h(\alpha)h(\gamma), \quad F_0^2 = F_x^2 + F_y^2, \quad m = M_z/a \\
&F_{01} = F_0\sin\psi_0, \quad F_{02} = F_0|\kappa|/\Delta, \quad F_{00}^2 = F_{01}^2 + F_{02}^2 \\
&\lambda_k = N_k\sigma(\alpha), \quad k = 1, 2; \quad \sigma_1(\gamma) = f_{12}g(\gamma) - f_{11}h(\gamma)
\end{aligned} \tag{3.5}$$

The following assertions hold.

Assertion 2. For static equilibrium of a rigid body, supported at two points on a rough surface with anisotropic dry friction, described by Φ_O, it is necessary and sufficient to satisfy condition (3.3), inequalities (3.4) and also the inequalities

$$|m| < \lambda_2, \quad |m - F_{01}| < \lambda_1, \quad F_{02} < \sqrt{\lambda_2^2 - m^2} + \sqrt{\lambda_1^2 - (m - F_{01})^2} \tag{3.6}$$

The quantities $\lambda_1, \lambda_2, m, F_{01}, F_{02}$ are defined by formulae (3.5).

Assertion 3.

1°. If $F_{00}^2 > (\lambda_1 + \lambda_2)^2$, inequalities (3.6) have no solutions and static equilibrium is impossible.
2°. If $F_{00}^2 > (\lambda_1 + \lambda_2)^2$, static equilibrium is only possible if $m \in [m_1, m_2]$, where m_1 and m_2 are calculated according to the following rules (we recall that $\lambda_2 \geq \lambda_1$, since, by definition, $N_2 \geq N_1$).

Suppose $F_{01} > 0$ (i.e., $\sin\psi_0 > 0$). Then

1) if $F_{00}^2 \geq \lambda_2^2 - \lambda_1^2 + 2\lambda_1 F_{01}$, then

$$m_{1,2} = \frac{1}{2F_{00}^2}\left(F_{01}\nu \pm F_{02}\sqrt{4\lambda_2^2F_{00}^2 - \nu^2}\right), \quad \nu = F_{00}^2 - \lambda_1^2 + \lambda_2^2 \tag{3.7}$$

2) if $\lambda_1^2 - \lambda_2^2 + 2\lambda_2 F_{01} \leq F_{00}^2 < \lambda_2^2 - \lambda_1^2 + 2\lambda_1 F_{01}$, then $m_1 = F_{01} - \lambda_1$, and m_2 is given by formulae (3.7), where we take the plus sign;
3) if $0 \leq F_{00}^2 < \lambda_1^2 - \lambda_2^2 + 2\lambda_2 F_{01}$, then

$$m_1 = F_{01} - \lambda_1, \quad m_2 = \begin{cases} \lambda_2 & \text{when } F_{01} > \lambda_2 - \lambda_1 \\ F_{01} + \lambda_1 & \text{when } 0 < F_{01} < \lambda_2 - \lambda_1 \end{cases}$$

Suppose $F_{01} < 0$ (i.e., $\sin\psi_0 < 0$). Then
4) if $F_{00}^2 \geq \lambda_2^2 - \lambda_j^2 - 2\lambda_1 F_{01}$, then m_1 and m_2 are given by formulae (3.7);
5) if $\lambda_1^2 - \lambda_2^2 - 2\lambda_2 F_{01} \leq F_{00}^2 < \lambda_2^2 - \lambda_1^2 - 2\lambda_1 F_{01}$, then $m_2 = F_{01} + \lambda_1$, and m_1 is given by formulae (3.7), where we take the minus sign;
6) if $0 \leq F_{00}^2 < \lambda_1^2 - \lambda_2^2 - 2\lambda_2 F_0$, then

$$m_2 = F_{01} + \lambda_1, \quad m_1 = \begin{cases} -\lambda_2 & \text{when } F_{01} < \lambda_1 - \lambda_2 \\ F_{01} - \lambda_1 & \text{when } \lambda_1 - \lambda_2 < F_{01} < 0. \end{cases}$$

Proof of Assertion 2. Without loss of generality we will assume that $\cos\alpha > 0$. We put $x = F_{2x}$. Then, we obtain from Eqs (3.1) and (3.2)

$$F_{1x} = -x - F_0\cos\gamma, \quad F_{1y} = -x\,\mathrm{tg}\,\alpha - F_0\sin\gamma + m/\cos\alpha, \quad F_{2y} = x\,\mathrm{tg}\,\alpha - m/\cos\alpha$$

Substituting these expressions into inequalities (1.6) ($k = 1, 2$) and using notation (3.5), we obtain the following inequalities, quadratic in x

$$Q_1(x) = x^2\sigma^2(\alpha) + 2xm[f_{12}g(\alpha) - f_{11}h(\alpha)] + m^2(f_{12}^2 + f_{11}^2) - \xi_2^2\cos^2\alpha \le 0 \tag{3.8}$$

$$Q_2(x) = Q_1(x) + 2F_0x\cos\alpha + (d_0 + \xi_2^2 - \xi_1^2)\cos^2\alpha \le 0 \tag{3.9}$$

where

$$d_0 = F_0^2\sigma^2(\gamma) + 2mF_0\sigma_1(\gamma)/\cos\alpha$$

and the remaining parameters are defined by formulae (3.5).

In order for equilibrium to be possible it is necessary and sufficient that inequalities (3.7) and (3.9) should have at least one common real solution x. Since the coefficient of x^2 in the expressions for Q_1 and Q_2 is positive, for this to be true it is necessary, first, that the quadratic trinomials $Q_1(x)$ and $Q_2(x)$ have only real roots $x_2 < x_1$ for $Q_1(x)$ and $x_4 < x_3$ for $Q_2(x)$, and second, that the segments $[x_2, x_1]$ and $[x_4, x_3]$ should have a non-empty intersection, i.e., the following two inequalities must be satisfied simultaneously

$$x_4 < x_1, \quad x_2 < x_3 \tag{3.10}$$

Algebraic calculations, which we omit here, give the following expressions for the roots $x_1, \ldots, x_4$, according to relations (3.8) and (3.9)

$$x_{1,2} = [-m\sigma_1(\alpha) \pm \sqrt{\Delta_1}]/\sigma^2(\alpha), \quad x_{3,4} = [-m\sigma_1(\alpha) - F_0\kappa\cos\alpha \pm \sqrt{\Delta_2}]/\sigma^2(\alpha) \tag{3.11}$$

where

$$\Delta_1 = \xi_2^2\sigma^2(\alpha)\cos^2\alpha - m^2\Delta^2\cos^2\alpha, \quad \xi_2^2 = \Delta^2N_2^2$$

$$\Delta_2 = \xi_1^2\sigma^2(\alpha)\cos^2\alpha - (m - F_0\sin\psi_0)^2\Delta^2\cos^2\alpha, \quad \xi_1^2 = \Delta^2N_1^2$$

Using formulae (3.11) it can be established that inequalities (3.10) and the conditions for the roots of the functions $Q_1(x)$ and $Q_2(x)$ to be real are equivalent to the inequalities

$$\Delta_1 > 0, \quad \Delta_2 > 0, \quad |F_0\kappa\cos\alpha| < \sqrt{\Delta_1} + \sqrt{\Delta_2}$$

which, after simple reduction and using notation (3.5), lead to relations (3.6).

Proof of Assertion 3. Consider the function

$$\psi(m) = \sqrt{\lambda_2^2 - m^2} + \sqrt{\lambda_1^2 - (m - F_{01})^2}$$

An investigation of this function shows that $\psi''_{m^2} < 0$, i.e., its graph is convex upwards, and it has a maximum at the point

$$m_* = F_{01}\lambda_2/(\lambda_1 + \lambda_2)$$

provided $|F_{01}| < \lambda_1 + \lambda_2$. But this inequality must necessarily be satisfied since only then can solutions of the first two inequalities (3.6) exist. In fact, we have from inequalities (3.6)

$$-\lambda_2 < m < \lambda_2, \quad -\lambda_1 + F_{01} < m < \lambda_1 + F_{01}$$

In order that these integrals should have a non-empty intersection, it is necessary and sufficient for the following conditions to be satisfied

$$-\lambda_2 < \lambda_1 + F_{01}, \quad -\lambda_1 + F_{01} < \lambda_2 \Rightarrow |F_{01}| < \lambda_1 + \lambda_2$$

Hence, the point $m = m_*$ belongs to the region in which the function $\psi(m)$ is defined, namely,

$$\max\psi(m) = \psi(m_*) = \sqrt{(\lambda_1 + \lambda_2)^2 - F_{01}^2}$$

We will investigate the roots of the equation $\psi(m) = F_{02}$. It is clear that when

$$F_{02} > \psi(m_*) = \sqrt{(\lambda_1 + \lambda_2)^2 - F_{01}^2}$$

the equation has no roots, i.e., when

$$F_{02}^2 + F_{01}^2 > (\lambda_1 + \lambda_2)^2$$

there are no solutions of inequalities (3.6), and equilibrium is impossible. The remaining part of Assertion 3 is proved by a direct check (this can be done most easily graphically).

Remark. For isotropic friction $\Phi_0 = fE$, and inequalities (3.6), from Assertion 2, have the form

$$F_0|\cos\psi_0| < \sqrt{f^2N_2^2 - m^2} + \sqrt{f^2N_1^2 - (m - F_0\sin\psi_0)^2}$$

and Assertion 3 gives the result obtained previously in Ref. 7 by the limit equilibrium method.

Example. We will use the results of Assertion 3 for the case when $N_1 = N_2 = N_0$. Then $\lambda_1 = \lambda_2 = \lambda_0 = N_0\sigma(\alpha)$ (see notation (3.5)). We will assume that $F_{01} > 0$, i.e., $\sin\psi_0 > 0$. Putting

$$\kappa_1 = |\kappa|/\Delta \Rightarrow \kappa_1^2 + \sin\psi_0 = \sigma^2(\alpha)\sigma^2(\gamma)/\Delta^2$$

we will have

$$F_{01} = F_0\sin\psi_0, \quad F_{02} = F_0\kappa_1, \quad F_{00}^2 = F_0^2(\kappa_1^2 + \sin^2\psi_0) = F_0^2\sigma^2(\alpha)\sigma^2(\gamma)/\Delta^2$$

By Assertion 3 only cases 1 or 3 are realized for equilibrium.
We will put

$$F_{11} = 2\lambda_0/\sqrt{\kappa_1^2 + \sin^2\psi_0}, \quad F_{12} = 2\lambda_0\sin\psi_0/(\kappa_1^2 + \sin^2\psi_0)$$

We then obtain the following result:

1) when $F_0 > F_{11}$ equilibrium is impossible;
2) when $F_0 \in [F_{12}, F_{11}]$ equilibrium is only possible when $m \in [m_1, m_2]$;

$$m_{1,2} = \frac{1}{2}\left(F_0\sin\psi_0 \pm k_1\sqrt{F_{11}^2 - F_0^2}\right)$$

3) when $0 \le F < F_{12}$ equilibrium is only possible when

$$m \in [m_1, m_2]; \quad m_1 = F_0\sin\psi_0 - \lambda_0, \quad m_2 = \lambda_0$$

It can be seen from these expressions that the boundary of the region of equilibrium in the plane of the parameters F_0, m for fixed ψ_0 and α consists of sections of straight lines and arcs of ellipses. We will illustrate this boundary for the principal moment m_c about the centre of the section A_1A_2. It is clear that

$$m_c = m - \frac{1}{2}F_0\sin\psi_0$$

We then obtain

$$\frac{1}{2}F_0\sin\psi_0 - \lambda_0 < m_c < -\frac{1}{2}F_0\sin\psi_0 + \lambda_0 \quad \text{when} \quad F_0 \in (0, F_{12})$$

$$-\frac{\kappa_1}{2}\sqrt{F_{11}^2 - F_0^2} < m_c < \frac{\kappa_1}{2}\sqrt{F_{11}^2 - F_0^2} \quad \text{when} \quad F_0 \in (F_{12}, F_{11})$$

In Fig. 3 we show the symmetrical region of equilibrium obtained in the plane of the parameters F_0, m_c for fixed values of the parameters ψ_0 and α. When the parameters ψ_0 and α change the region of equilibrium also changes. Assuming that the friction is orthotropic, i.e., $f_{12} = f_{21} = 0$, we will consider the two simplest cases.

The case when $\psi_0 = 0$. Then $F_{12} = 0$, i.e., the boundaries of the regions of equilibrium when $\alpha \in (0, \pi/2)$ consists solely of arcs of ellipses (the left side of Fig. 4) with semiaxes

$$F_{11} = 2N_0f_{11}f_{22}/f_0, \quad m_{c_1} = N_0f_0; \quad f_0 = \sqrt{f_{22}^2\cos^2\alpha + f_{11}^2\sin^2\alpha}$$

It can be seen that the lengths of the semiaxes of the ellipses depend considerably on the angle α – the slope of the section A_1A_2 to the Ox axis, where the length of one of the semiaxes increases monotonically, while the length of the other decreases as the angle α changes.

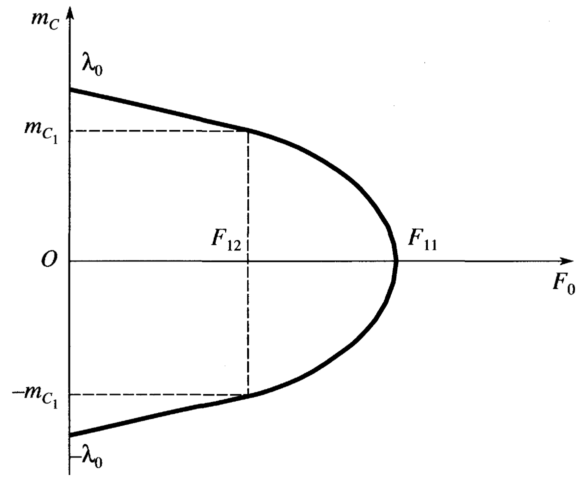

Fig. 3.

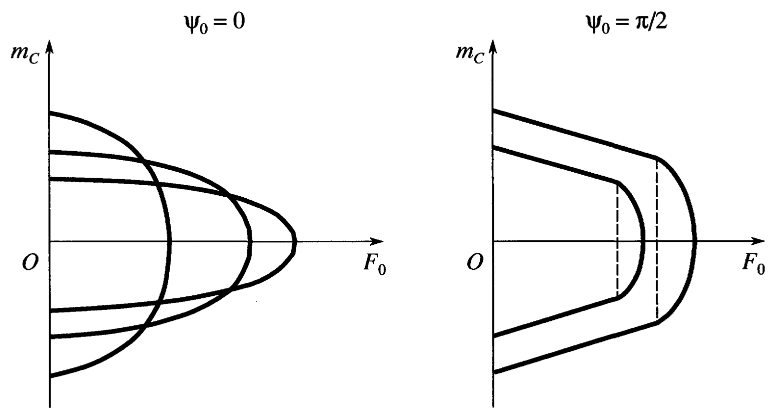

Fig. 4.

The case when $\psi_0 = \pi/2$. Then

$$F_{12} = 2N_0 f_{11}^2 f_{22}^2/(f_0 f_{00}^2), \quad \lambda_0 = N_0 f_0, \quad F_{11} = 2N_0 f_{11} f_{22}/f_{00};$$
$$f_{00} = \sqrt{f_{22}^2 \sin^2\alpha + f_{11}^2 \cos^2\alpha}$$

Here the regions of equilibrium for different α consist of sections of straight lines and arcs of ellipses smoothly joined to them (the right side of Fig. 4), and, since F_{11} and λ_0 are simultaneously monotonically decreasing or increasing functions of the angle α, these regions are imbedded in one another.

Note that the regions of equilibrium represented in Fig. 4 differ considerably from the analogous regions of equilibrium obtained previously (Ref. 1, p.227).

4. Formulation and justification of the results for the case when n = 3

Suppose $A_j(x_j,y_j)$ ($j = 1, 2, 3$) is a support point (Fig. 5). The origin of coordinates O coincides with the point A_1. The anisotropic friction is specified by the positive definite 2×2 matrix $\Phi_0 = ||f_{ij}||$ of the supporting rough plane in the Oxy axes.

We will assume that non-negative normal reactions N_j are specified at the support points A_j, which are found from equilibrium equations (1.1) and depend only on the external specified forces, their moments and geometrical parameters of the body. Suppose F_j are the static friction forces at the points A_j. Then the equilibrium equations have the form

$$F_{1x} = -(F_{2x} + F_{3x} + F_x), \quad F_{1y} = -(F_{2y} + F_{3y} + F_y)$$
$$M_z = F_{2x}y_2 - F_{2y}x_2 + F_{3x}y_3 - F_{3y}x_3 \tag{4.1}$$

with constraints (1.6) for the static friction forces.

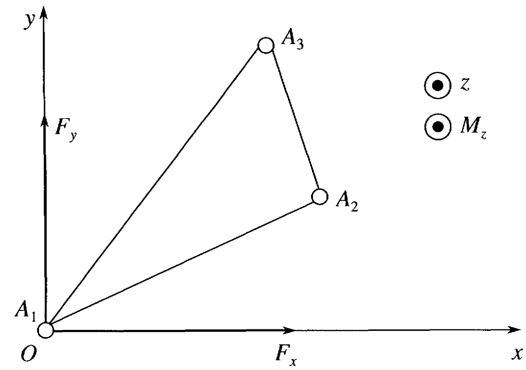

Fig. 5.

We introduce the matrix Φ_1 and the variables u_j, $\upsilon_j (j = 1, 2, 3)$, P_1, P_2 using the formulae

$$\Phi_1 = \begin{Vmatrix} f_{22} & -f_{12} \\ f_{21} & -f_{11} \end{Vmatrix}, \quad \begin{Vmatrix} u_j \\ \upsilon_j \end{Vmatrix} = \Phi_1 \begin{Vmatrix} F_{jx} \\ F_{jy} \end{Vmatrix}, \quad j = 1, 2, 3, \quad \begin{Vmatrix} P_1 \\ P_2 \end{Vmatrix} = \Phi_1 \begin{Vmatrix} F_x \\ F_y \end{Vmatrix}$$

Using the above notation and formulae (4.1) for the principal moment M_z we obtain the expression

$$\Psi = M_z\Delta = u_2a_2 + \upsilon_2b_2 + u_3a_3 + \upsilon_3b_3 \tag{4.2}$$

where

$$a_j = f_{11}y_j - f_{21}x_j, \quad b_j = -f_{12}y_j + f_{22}x_j, \quad j = 2, 3; \quad \Delta = f_{11}f_{22} - f_{12}f_{21} > 0$$

Constrains (1.6) take the form

$$(u_2 + u_3 + P_1)^2 + (\upsilon_2 + \upsilon_3 + P_2)^2 \le \xi_1^2 \tag{4.3}$$

$$u_2^2 + \upsilon_2^2 \le \xi_2^2, \quad u_3^2 + \upsilon_3^2 \le \xi_3^2 \tag{4.4}$$

The problem now is to determine the extrema of the function Ψ (4.2) with constraints (4.3) and (4.4). Solving this problem we obtain the condition for static equilibrium to be possible in terms of the parameters M_z, P_1, P_2, ξ_2, ξ_3, a_2, b_2, a_3, b_3 which are also easy to formulate for the initial parameters F_x, F_y, x_2, y_2, x_3, y_3.

Suppose Q_1 and Q_2, Q_3, are sets (convex) in the four-dimensional space $\{u_2, \upsilon_2, u_3, \upsilon_3\}$, defined by inequalities (4.3) and (4.4) respectively. Then, in view of the linearity of the function Ψ, its maximum and minimum are only reached on the boundary of intersection of the sets Q_1, Q_2 and Q_3. Hence, the search for an extremum consists of inspecting all the versions of the boundaries of intersection of these sets. We will consider these possible cases.

Case 1. Suppose the set $D_1 = Q_1 \cap Q_2 \cap Q_3$ is non-empty. We will investigate the values of the function Ψ on its boundary.

The following assertion holds.

Assertion 4.

1°. The set Γ_1 of points of the boundary of the set D_1, at which inequalities (4.3) and (4.4) become equalities, is non-empty if and only if

$$P = \sqrt{P_1^2 + P_2^2} \in [\xi_{01}, \xi_{02}] \tag{4.5}$$

where

$$\xi_{01} = \max\{\xi_1 - \xi_2 - \xi_3, \xi_2 - \xi_1 - \xi_3, \xi_3 - \xi_1 - \xi_2\}, \quad \xi_{02} = \xi_1 + \xi_2 + \xi_3$$

2°. The function Ψ (4.2) in the set Γ_1 is given by the formula

$$\Psi(\varphi) = \xi_2(a_2\cos\varphi + b_2\sin\varphi) + \frac{1}{q_2}(zq_1 \pm q_0\sqrt{\xi_3^2q_2 - z^2}) \tag{4.6}$$

where

$$q_2 = 1 + \frac{P^2}{\xi_2^2} + \frac{2}{\xi_2}(P_1\cos\varphi + P_2\sin\varphi)$$

$$q_1 = a_3\cos\varphi + b_3\sin\varphi + \frac{1}{\xi_2}(a_3P_1 + b_3P_2)$$

$$q_0 = a_3\sin\varphi - b_3\cos\varphi + \frac{1}{\xi_2}(a_2P_2 - b_3P_1)$$

$$z = h_0 - P_1\cos\varphi - P_2\sin\varphi, \quad h_0 = \frac{1}{2\xi_2}(\xi_1^2 - \xi_2^2 - \xi_3^2 - P^2) \tag{4.7}$$

and the parameter φ varies so that

$$P_1\cos\varphi + P_2\sin\varphi \in [P_{01}^-, P_{01}^+]; \quad P_{01}^{\pm} = \frac{(\xi_1 \pm \xi_3)^2 - \xi_2^2 - P^2}{2\xi_2}$$

(when inclusion (4.5) is satisfied the set of variations of the parameter φ is non-empty!)

Hence, the initial problem of an extremum here reduces to investigating the roots of the equation $d\Psi/d\varphi = 0$, the function $\Psi(\varphi)$ is given by formulae (4.6) and (4.7), while the parameter φ varies within the limits indicated.

The proof consists of solving Eqs. (4.2) and (4.3), (4.4) simultaneously (where the inequalities are replaced by equalities). Suppose $u_2 = \xi_2 \cos\varphi$, $\upsilon_2 = \xi_2 \sin\varphi$. Then the first equality of (4.4) is satisfied. From the remaining three equalities we have

$$a_3u_3 + b_3\upsilon_3 = \Psi - \xi_2(a_2\cos\varphi + b_2\sin\varphi)$$

$$\left(\cos\varphi + \frac{P_1}{\xi_2}\right)u_3 + \left(\sin\varphi + \frac{P_2}{\xi_2}\right)\upsilon_3 = h_0 - (P_1\cos\varphi + P_2\sin\varphi) \tag{4.8}$$

$$u_3^2 + \upsilon_3^2 = \xi_3^2 \tag{4.9}$$

The quantity h_0 is defined by the last formula of (4.7).

We solve the linear system (4.8) for u_3 and v_3 and substitute the results into Eq. (4.9). We obtain the following quadratic equation in $\Psi_1 = \Psi - \xi_2(a_2\cos\varphi + b_2\sin\varphi)$

$$\Psi_1^2 q_2 - 2\Psi_1 z q_1 + (a_3^2 + b_3^2)z^2 - \xi_3^2 q_0^2 = 0 \tag{4.10}$$

The quantities q_1, q_2, q_3 and z are defined by (4.7). Using the identity

$$q_1^2 - q_2(a_3^2 - b_3^2) = -q_0^2$$

we obtain expressions (4.6) from Eq. (4.10).

The correctness of the first part of Assertion 4 follows from the condition for the radicand in (4.6) to be positive.

In fact

$$\xi_3^2 q_2 - z^2 = \xi_3^2\left[1 + \frac{P^2}{\xi_2^2} + \frac{2h_0 - 2z}{\xi_2}\right] - z^2 \geq 0$$

Consequently,

$$z \in [z_1^+, z_1^-]; \quad z_1^{\pm} = -\frac{\xi_3}{\xi_2}(\pm\xi_1 + \xi_3)$$

Moreover, by the penultimate equality of (4.7)

$$z \in (h_0 - P, h_0 + P)$$

which leads to inclusion (4.5).

Case 2. We will investigate the function Ψ on the boundary of the set $D_2 = Q_i \cap Q_j$ $(i \neq j)$, where the points of this boundary must belong to the interior of the set Q_k $(k \neq i, k \neq j)$. If the function Ψ has local-extremum points in this set, these points are also global extrema by virtue of its linearity.

Assertion 5. Consider the boundary Γ_2 of the set $D_2 = Q_2 \cap Q_3$, i.e., inequalities (4.4) become equalities. Then the function Ψ has a local extremum on Γ_2, which is an internal point of the set Q_1 solely when

$$\pm\left(\cos A_{23} + \frac{P}{\xi_3}\cos\gamma_2 + \frac{P}{\xi_2}\cos\gamma_3\right) < \frac{\xi_1^2 - \xi_2^2 - \xi_3^2 - P^2}{2\xi_2\xi_2} \tag{4.11}$$

where the plus sign is taken for a maximum point and a minus sign is taken for a minimum point, A_{23} is the angle between the vectors $\mathbf{d}_2 = (a_2, b_2)$ and $\mathbf{d}_3 = (a_3, b_3)$, and γ_2 and γ_3 are the angles which the vector $\mathbf{P} = (P_1, P_2)$ makes with the vectors $\mathbf{d}_2$ and $\mathbf{d}_3$.

Proof. The boundary Γ_2 is given by the formulae

$$u_j = \xi_j \cos\varphi_j, \quad \upsilon_j = \xi_j \sin\varphi_j; \quad j = 2, 3, \quad \varphi_j \in (0, 2\pi)$$

Then the extremum points of the function Ψ from (4.2) are specified by the equalities

$$u_j = \pm\xi_j \frac{a_j}{\sqrt{a_j^2 + b_j^2}}, \quad \upsilon_j = \pm\xi_j \frac{b_j}{\sqrt{a_j^2 + b_j^2}}; \quad j = 2, 3$$

Substituting these expressions into inequality (4.3), we obtain inequality (4.11).

Inequalities similar to (4.11) are obtained for the sets $Q_1 \cap Q_3$ and $Q_1 \cap Q_2$ by cyclic permutation of the subscripts.

Case 3. The set $Q_1 \cap Q_2 \cap Q_3$ is empty, i.e. equilibrium is certainly impossible.

Assertion 6. Inequalities (4.3)–(4.5) have no solutions if and only if

$$P > \xi_{02} = \xi_1 + \xi_2 + \xi_3 \tag{4.12}$$

Proof. It is clear that inequalities (4.3) and (4.4) have no common solution if and only if

$$\Phi = \min[(u_2 + u_3 + P_1)^2 + (\upsilon_2 + \upsilon_3 + P_2)^2] > \xi_1^2$$

under conditions (4.4). Using, for example, the method of Lagrange undetermined multipliers, we can obtain that

$$\Phi = \begin{cases} (P - \xi_2 - \xi_3)^2 & \text{when} \quad P > \xi_2 + \xi_3 \\ 0 \quad \text{when} \quad 0 < P < \xi_2 + \xi_3 \end{cases}$$

Hence, if $P > \xi_2 + \xi_3$, inequalities (4.3) and (4.4) are incompatible when condition (4.12) is satisfied.

Case 4. The set Q_1 contains an intersection of the sets Q_2 and Q_3, i.e., $Q_1 \supset [Q_2 \cap Q_3]$. Other possible similar situations are obtained cyclically by permutation of the subscripts.

Assertion 7. Suppose that, among the numbers

$$\xi_1 - \xi_2 - \xi_3, \; \xi_2 - \xi_1 - \xi_3, \; \xi_3 - \xi_1 - \xi_2 \tag{4.13}$$

there is one positive number, for example, the first of them (there cannot, obviously, be two positive numbers when $\xi_j > 0$, j = 1, 2, 3). Then, if $0 \le P < \xi_1 - \xi_2 - \xi_3$ we have $Q_1 \supset [Q_2 \cap Q_3]$, and the extremum of Ψ is calculated in the same way as in Assertion 5 (the extremum point necessarily belongs to the interior of the set Q_1). Other possible situations are obtained by cyclic permutation of the subscripts.

Proof. It is clear that the inclusion $Q_1 \supset [Q_2 \cap Q_3]$ is only satisfied when

$$\Phi = \max[(u_2 + u_3 + P_1)^2 + (\upsilon_2 + \upsilon_3 + P_2)^2] < \xi_1^2$$

for conditions (4.4). The values of Φ can be calculated, for example, by the method of Lagrange undetermined multipliers. We have

$$\Phi = (P + \xi_2 + \xi_3)^2 < \xi_1^2 \Rightarrow P < \xi_1 - \xi_2 - \xi_3$$

Example. We will consider the application of the results obtained to the case when $F_x = F_y = 0$. Then, $P_1 = P_2 = P = 0$.

We must distinguish two cases.

1°. The quantities ξ_1, ξ_2, ξ_3 do not satisfy the triangle inequalities. Then among the numbers (4.13) there is one positive number and we can use the result of Assertion 7. Suppose, for example, that $\xi_1 - \xi_2 - \xi_3 > 0$. Then

$$\Psi_{\text{extr}} = M_z\Delta = \pm(\xi_2\sqrt{a_2^2 + b_2^2} + \xi_3\sqrt{a_3^2 + b_3^2})$$

The remaining possible situations are obtained by cyclic permutation of the subscripts. For example, if $\xi_2 - \xi_1 - \xi_3 > 0$, we have

$$\Psi_{\text{extr}} = M_z\Delta = \pm(\xi_1\sqrt{a_2^2 + b_2^2} + \xi_3\sqrt{(a_2 - a_3)^2 + (b_2 - b_3)^2})$$

2°. The quantities ξ_1, ξ_2, ξ_3 satisfy the triangle inequalities. Then, all the numbers (4.13) are negative, and Assertions 4 and 5 can be used. We must distinguish two cases here. If inequality (4.11) is satisfied (or an inequality similar to it by cyclic permutation of the subscripts), an extremum of the function Ψ can be found in the same way as in Assertion 7 (with appropriate cyclic permutation of the subscripts). If inequalities (4.11) are not satisfied, i.e.,

$$\pm\cos A_{23} > \frac{\xi_1^2 - \xi_2^2 - \xi_3^2}{2\xi_2\xi_3} \ (1\ 2\ 3)$$

(A_{ij} is the angle between the vectors $\mathbf{d}_i$ and $\mathbf{d}_j$, (1 2 3) denotes cyclic permutation of the subscripts) we must use Assertion 4, which in this case reduces the search for an extremum to an investigation of the function

$$\Psi(\varphi) = \Psi_c \cos\varphi + \Psi_s \sin\varphi \tag{4.14}$$

where

$$\Psi_c = \xi_2 a_2 + h_0 a_3 \mp b_3\sqrt{\xi_3^2 - h_0^2}, \quad \Psi_s = \xi_2 b_2 + h_0 b_3 \pm a_3\sqrt{\xi_3^2 - h_0^2}$$

$$h_0 = \frac{\xi_1^2 - \xi_2^2 - \xi_3^2}{2\xi_2}, \quad \varphi \in (0, 2\pi)$$

It is clear that $|h_0| < \xi_3$, since this inequality reduces to inequalities for a triangle with sides ξ_1, ξ_2, ξ_3. Hence, the function $\Psi(\varphi)$ (4.14) is correctly defined over the whole section $\varphi \in [0, 2\pi]$. The extrema of the function $\Psi(\varphi)$ are given by the formulae

$$\Psi_{\text{extr}} = \pm\sqrt{\Psi^2}$$

$$\Psi^2 = \Psi_c^2 + \Psi_s^2 = \tilde{\Psi}^2 + 2|b_2 a_3 - b_3 a_2|\xi_2\left[\left(\xi_3 - \frac{\xi_1^2 - \xi_2^2 - \xi_3^2}{2\xi_2}\right)\left(\xi_3 + \frac{\xi_1^2 - \xi_2^2 - \xi_3^3}{2\xi_2}\right)\right]^1$$

$$\tilde{\Psi}^2 = \frac{1}{2}[\xi_1^2(l_2^2 + l_3^2 - l_1^2) + \xi_2^2(l_1^2 + l_2^2 - l_3^2) + \xi_3^2(l_3^2 + l_1^2 - l_2^2)]$$

$$l_1^2 = (a_2 - a_3)^2 + (b_2 - b_3)^2, \quad l_2^2 = a_2^2 + b_2^2, \quad l_3^2 = a_3^2 + b_3^2$$

Using Heron's formula for a triangle with sides ξ_1, ξ_2, ξ_3 and the fact that $|b_2 a_3 - b_3 a_2| = 2S_{ab}$, where S_{av} is the area of the triangle formed by the vectors $\mathbf{d}_2 = (a_2, b_2)$ and $\mathbf{d}_3 = (a_3, b_3)$, we obtain the following result

$$\Psi^2 = \tilde{\Psi}^2 + 8S_{ab}S$$

where S is the area of the triangle with sides ξ_1, ξ_2, ξ_3.

Remarks. The results obtained depend very much on the orientation of the triangle $A_1A_2A_3$ and of the principal vector $\mathbf{F}_{xy}$ with respect to the axes of coordinates Oxy, since the matrix of anisotropic friction Φ_0 depends on the choice of the system of coordinates Oxy. The O*x* and O*y* axes can be directed along the principal axes of the matrix Φ_0, and if they are mutually perpendicular we will have orthotropic friction, and the equilibrium condition will also depend both on the orientation of the triangle $A_1A_2A_3$ and on the vector $\mathbf{F}_{xy}$ with respect to these principal axes.

Similar results were obtained in Ref. 7 for isotropic friction by the limit equilibrium method. The problem of guaranteed equilibrium on a plane with orthotropic friction was considered in Ref. 4 for an arbitrary number of support points.

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